

Dynamic Programming Algorithms for Lot-Sizing Problems

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Abstract

Lot-sizing problem has been extensively researched in many aspects. In this review, we summarize key points of dynamic programming algorithms in lot-sizing problems and give some hints helping further designing algorithms for more variations of lot-sizing problems ([4]).

Keywords: Lot-sizing, Dynamic programming, Minimum cost flows

1. Introduction and Notations

For lot-sizing mixing integer problems, tight description of their polyhedra has been achieved for many variations by cutting plans, extended formulations, totally unimodular matrices and other properties crossing wide fields, see [2]. Dynamic programming (DP) algorithms are still powerful for general lot-sizing problems because their structures. In this review, we summarize key points of dynamic programming algorithms in lot-sizing problems and give some hints helping further designing algorithms for more variations of lot-sizing problems.

First we give notations and formulate the general lot-sizing problem as follows. Let n be the length of the planning time horizon. For each period $t \in \{1, 2, \dots, n\}$ the following data are given:

- p_t unit production cost in t ,
- h_t unit holding cost in t (defined also for $t = 0$),
- q_t set-up cost in t ,
- d_t demand in t ,
- C_t production capacity in t .

We suppose all data is nonnegative rational. For easy of presentation, we also denote $d_{kl} = \sum_{t=k}^{t=l} d_t$.

And variables are defined as follows:

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- x_t production in t ,
- s_t stock at the end of t (defined also for $t = 0$),
- y_t set-up binary variables,

Now we can formulate lot-sizing model (LS):

$$\min \sum_{t=1}^n (p_t x_t + h_t s_t + q_t y_t) \tag{1.1}$$

$$s_{t-1} + x_t = d_t + s_t \quad \text{for } 1 \leq t \leq n \tag{1.2}$$

$$x_t \leq C_t y_t \quad \text{for } 1 \leq t \leq n \tag{1.3}$$

$$x_t, s_t \in \mathbf{R}_+, y_t \in \{0, 1\} \quad \text{for } 1 \leq t \leq n. \tag{1.4}$$

Throughout the manuscript we suppose $s_0 = 0$ for simplicity.

When the production capacities are periods varying, the lot-sizing problem is NP-hard. The rest of the manuscript is consisted as follows, in Section 2, we consider the problem with unbounded capacities ([2]). And constant capacities problem is described in Section 3 ([2]). In final section, we consider the problem with production/delivery time windows ([1], [3]).

2. Unbounded Capacities

When lot-sizing problem is unbounded capacities, or uncapacitated (LS-U), equations (1.3) are rewritten as

$$x_t \leq M y_t \quad \text{for } 1 \leq t \leq n, \tag{2.1}$$

where M is a large positive number.

Suppose that set-up variables $y \in \{0, 1\}^n$ are known, the flow conservation of constraints (1.2) together with set-up conditions (2.1) can be represented as flows in a network. In Figure 2.1., such a network with $n = 5$ is shown as an instance.

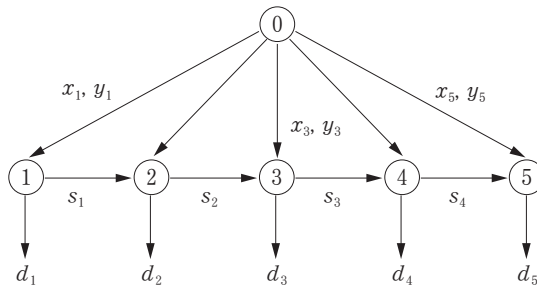


Figure 2.1. The fixed charge network flow variables

Now it is a natural way to view LS-U as a minimum cost flow problem of a network. A well-known and fundamental property of minimum cost network flow problem tells:

Observation 2.1: For a basic (or extreme) feasible solution of a minimum cost network flow problem,

the arcs corresponding to variables with flows strictly between their lower and upper bounds form an acyclic graph.

From above observation, we immediately have an important property about the structure of optimal solutions to $LS-U$.

Proposition 2.1: *There exists an extreme optimal solution to $LS-U$ in which $s_{t-1}x_t = 0$ for all t . More precisely, a subset of periods $1 \leq t_1 < \dots < t_r \leq n$ in which production takes place. The amount produced in t_j is $d_{t_j} + \dots + d_{t_{j+1}-1}$ for $j = 1, \dots, r$ with $t_{r+1} = n+1$.*

The intervals $[t_1, t_2-1], [t_2, t_3-1], \dots$ of a solution with no stock entering or leaving the interval, are called *regeneration intervals*. With this notation concept, we can say weakly a decomposition property that is a necessary condition for dynamic programming algorithm.

Corollary 2.2: *There exists an extreme optimal solution, such that periods are partitioned into sub-intervals, demands are not splitting cross these intervals (regeneration intervals).*

Using the flow balance equalities (1.2), we can omit stock valuables from object function. Without loss of generality, we take $\sum_{t=1}^n (p_t x_t + q_t y_t)$ as objective function in the following.

Now we can give a dynamic programming.

Let $G(t)$ be the minimum cost of solving the problem over the first t periods, and let $\phi(k, t)$ be the minimum cost of solving the problem over the first t periods subject to the additional condition that the last production periods is k for some $k \leq t$. From the definition we have

$$G(t) = \min_{k:k \leq t} \phi(k, t).$$

By Proposition 2.1 and also *Principle of Optimality* for DP, $\phi(k, t)$ can be calculated as

$$\phi(k, t) = G(k-1) + q_k + p_k d_{kt}.$$

Dynamic programming recursion for $LS-U$

$$G(0) = 0$$

$$G(t) = \min_{k:k \leq t} [G(k-1) + q_k + p_k d_{kt}] \quad \text{for } t = 1, \dots, n.$$

Finally, note that DP is initial based on the fact $s_{t-1}x_t = 0$ for $1 \leq t \leq n$, although s_t is omitted in DP, the property is remained in the form of generation intervals, and see also Corollary 2.2.

3. Constant Capacities

As same as Section 2, lot-sizing problem with constant capacities ($LS-CC$), equations (1.3) are rewritten as

$$x_t \leq Cy_t \quad \text{for } 1 \leq t \leq n. \quad (3.1)$$

The concept of regeneration intervals is same as $LS-U$, within a regeneration interval, with the same arguments, there is at most one period in which the production $x_p \notin \{0, C\}$. Note this property is also applied to the case when capacities are time varying.

Now we go on to DP for the problem of finding an extreme optimal solution on a given regeneration interval $[k, l]$. Let $\rho_{kl} = d_{kl} - \left\lfloor \frac{d_{kl}}{C} \right\rfloor C$ with $0 \leq \rho_{kl} < C$. Let $G_k(t, \tau, \theta)$ be the value of a minimum cost solution for periods k up to t during which production occurs τ times at full capacity and $\theta \in \{0, 1\}$ times at level ρ_{kl} . We omit infeasible cases $\tau C \leq d_{kl}$ and $\tau C + \rho_{kl} \leq d_{kl}$ (or set to an arbitrary large number) for simplicity.

Dynamic programming recursion for regeneration intervals of $LS-CC$

Starting from $G_k(k, 1, 0) = q_k + p_k C$ and $G_k(k, 0, 1) = q_k + p_k \rho_{kl}$, $G_k(k, 0, 0) = 0$ if $d_k = 0$ and $G_k(k, \tau, \theta) = \infty$ otherwise.

$$G_k(t, \tau, 0) = \min \begin{cases} G_k(t-1, \tau, 0) \\ G_k(t-1, \tau-1, 0) + q_t + p_t C \end{cases} \quad (3.2)$$

for $t = k, \dots, l, \tau = 0, \dots, \left\lfloor \frac{d_{kl}}{C} \right\rfloor$

$$G_k(t, \tau, 1) = \min \begin{cases} G_k(t-1, \tau, 1) \\ G_k(t-1, \tau-1, 1) + q_t + p_t C \\ G_k(t-1, \tau, 0) + q_t + p_t \rho_{kl} \end{cases} \quad (3.3)$$

for $t = k, \dots, l, \tau = 0, \dots, \left\lfloor \frac{d_{kl}}{C} \right\rfloor$

Where, fix initially $G_k(k-1, \tau, \theta) = 0$ for all (τ, θ) with $\tau \leq 0$ and $\theta \leq 0$, and $G_k(k-1, \tau, \theta) = \infty$ otherwise.

Note when $\rho = 0$ it suffices to use the recursion for $G_k(t, \tau, 0)$ only.

With the optimal solutions on all generation intervals, $LS-CC$ can be solved with shortest path problem.

4. Delivery/Production Time Windows

Suppose that we have K demands to be satisfied over n periods, where $[e_i, l_i]$ is the time window of demand $i = 1, \dots, K$, and e_i and l_i are called the *earliest* and *latest* delivery (production) period of demand i , respectively. In the case of production time window, delivery of the order takes place in period l_i . Add to the parameters and variables in Section 1, we also define:

P_{it} unit production cost plus holding cost to satisfy demand $i = 1, \dots, K$ by the production in period $t = 1, \dots, l_i$,

x_{it} the amount of demand $i = 1, \dots, K$ produced in period $t = 1, \dots, n$.

Demands are indexed according to their earliest delivery/production periods (i.e., $e_i \leq e_j$ for

all $i < j$), it is also known that $K \leq 2n - 1$ ([3]).

The relation of P_{it} with traditional cost setting can be formulated as follows:

$$P_{it} = \begin{cases} p_t + \sum_{k=t}^{e_i-1} h_k & \text{if } t < e_i \\ p_t & \text{if } e_i \leq t \leq l_i \end{cases}$$

under delivery time windows, and

$$P_{it} = \begin{cases} \infty & \text{if } t < l_i \\ p_t + \sum_{k=t}^{l_i-1} h_k & \text{if } e_i \leq t \leq l_i \end{cases}$$

under production time windows.

Combine with new notations, we reformulate the lot-sizing problem with time windows as follows (*LS-TW*).

$$\begin{aligned} & \min \sum_{t=1}^n q_t y_t + \sum_{i=1}^K \sum_{t=1}^{l_i} P_{it} x_{it} \\ & \sum_{t=1}^{l_i} x_{it} = d_i \quad \text{for } 1 \leq i \leq K \\ & \sum_{i:t \leq l_i} x_{it} \leq C_t y_t \quad \text{for } 1 \leq t \leq n \\ & x_{it} \in \mathbf{R}_+ \text{ for } 1 \leq i \leq K; \text{ for } 1 \leq t \leq l_i \\ & y_i \in \{0, 1\} \quad \text{for } 1 \leq i \leq K. \end{aligned}$$

4.1. Uncapacitated production time windows

With uncapacitated production time windows (*LS-U-TWP*), we have following property.

Observation 4.1: *There exists an optimal solution in which each order (or demand) i is produced in a single period.*

Note this property is more strong than the case of *LS-CC*. By the property we have immediately:

Let $H(t, k)$ be the value of an optimal solution for periods $1, \dots, t$ in which the demands d_1, \dots, d_k are produced in or before period t .

Let $G(t, k)$ be the value of an optimal solution for periods $1, \dots, t$ in which the demands d_1, \dots, d_k are produced in or before period t and d_k is produced in t .

The above definitions being possible implies: i) $l_i \leq l_j$ for all $i < j$, i.e., it is noninclusive. ii) With noninclusive time windows, there exists an optimal solution in which order i is produced before (or at the same time) as order j for all $i < j$.

Dynamic programming recursion for LS—U—TWP

$$H(t, k) = \min \begin{cases} H(t-1, k) \\ G(t, k) \end{cases} \quad (4.1)$$

for t, k with $e_k \leq t$

$$G_k(t, k) = \min \begin{cases} H(t-1, k-1) + q_t + p_t d_k \\ G(t, k-1) + p_t d_k \end{cases} \quad (4.2)$$

for t, k with $t \in [e_k, l_k]$

4.2. Constant capacitated time windows

To obtain the property similar to regeneration interval (i.e., decomposition condition required by *DP*), some restrictions on cost parameters are assumed. With $P_{is} \geq P_{it}$ for $s < t$, we have:

Observation 4.2: *There exists an optimal solution of the LS—CC—TW such that if period t is a partial production period, then any demand i with $l_i \geq t$ is supplied only by productions in periods not earlier than t .*

Let $C(s, t)$, $1 \leq s < t \leq n+1$, as the minimum cost of satisfying all demands i with $s \leq l_i < t$ by production during the periods $s, s+1, \dots, t-1$, where s is a (full or partial) production period and k with $s < k < t$, has either no production or full production.

Let $F(t)$, $1 \leq t \leq T+1$ be the minimum cost of satisfying all demands i with $1 \leq l_i < t$ by production during the periods $1, \dots, t-1$.

Now we have following recursive equations (more precisely, decomposition).

$$F(1) = 0$$

$$F(t) = \min_{1 \leq s < t} [F(s) + C(s, t)] \quad \text{for } t = 2, \dots, n+1.$$

Note, in above recursion formulas, no details on how to compute $C(s, t)$, which is further based on following assumption and observation.

By $P_{is} + P_{jt} \leq P_{js} + P_{it}$ for $s < t$ and $i < j$, we can obtain:

Observation 4.3: *There exists an optimal solution of the LS—CC—TW such that if the demands i is supplied by production in period s , then any demand $j > i$ with $s < l_i < t$ is supplied only by production during $[s, t-1]$.*

To complete the algorithm, $C(s, t)$ needed to be decomposed further, especially state of s is divided by several cases. It is technical and too detail to be described here.

References

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