Ordered Weighted Optimization related to Majorization

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Abstract

Majorication is a classical subject in mathematics. Optimization related to majorization have also been researched ([2] [6]). Recently, majorization for partially ordered sets has been published ([1]). On the other hand, ordered weights optimization problems, known as OWA (Ordered Weighted Average), and OM (Ordered Median) are researched independently ([3]). The keyword between these two themes is order. The manuscript is: after a review, discussing the relation between them.

Keywords: principle ideal, partial ordered set and ideal, majorization, optimization

1. Introduction

Majorization is a classical subject in mathematics, a well known result related to double stochastic matrices and permutation matrices is revealed by Birkhoff in 1946. Optimizations related to majorization and its partial ordered generalization are also developed, for example, see [2] and [6]. On the other hand, ordered weights optimization problems, known as OWA(Ordered Weighted Average), and OM(Ordered Median) are researched independently ([3]). The complexes of these optimization problems vary from trivial case ([2]) to NP-hard ([3]) based on their structures. Our motivation is investigating the relation between the ordered weight problems and their underlined structures. One key factor of structure is symmetry. Problems related to majorization usually guarantee some symmetries, while values in problems titled with OWA or OM vary with permutation. The manuscript is a first step, mainly in a review stage and searching for possible research direction.

There are a lot of applications related to these concepts. Majorization as a Lorens curve (1906) ([5]) can be used as a measure of economics distribution or wealth inequality, which is also global hot topic now. Majorization for partial ordered sets can be applied to source allocation ([1]). OWA and OM are formulations for facilities allocation. Some special weights for OWA or OM are also unication description of fundamental statistics in data analysis, which we will mention in next section.

The manuscript is organized as follows. After basic concepts introduction in next section, in Section 3 we treate problems related to special ordered set, principle majorization ideals ([2]). We discuss general ordered sets and related majorization problems in Section 4 ([1]). In final section, we show general ordered weighted average problems.

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2. Denitions and preliminary

We begin with two basic concepts.

For $x, y \in \mathbf{R}^n$ we say that x is *majorized* by y, and write $x \preceq y$, if

$$\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} y_{[j]} \qquad \text{for } k = 1,\ 2, \cdots, n-1, \tag{2.1}$$

$$\sum_{j=1}^{n} x_{[j]} = \sum_{j=1}^{n} y_{[j]}, \tag{2.2}$$

here $x_{[j]}$ denotes the jth largest component of x.

Given a weight $w = (w_1, w_2, \cdots, w_n)$, data or function $x = (x_1, x_2, \cdots, x_n)$, an OWA or OM formulation here is defined as maximization or minimization of the following

$$\sum_{i=1}^{n} w_i x_i \tag{2.3}$$

$$s.t. \quad x_1 \geq x_2 \geq \cdots \geq x_n. \tag{2.4}$$

Note, let $w = (1, 0, \cdots, 0)$, or $w = (1/n, 1/n, \cdots, 1/n)$, we obtain minimum or average of x, maximum or median of x can been done in similar ways.

3. Principle majorization ideals

Now we introduce the relation between majorization and OWA.

Let

$$\mathbf{D}^n = \{\, x \in \mathbf{R}^n \ : \ x_1 \geq x_2 \ \geq \cdots \geq x_n \,\}.$$

Given a vector $b \in \mathbf{R}^n$, a polytope majorized by b on $\mathbf{D}^n$ is dened as

$$M(b) = \{\, x \in \mathbf{D}^n \ : \ x \preceq b \,\}. \tag{3.1}$$

As indicated in [2], majorization is reflexive and transitive on $\mathbf{R}^n$, it is also antisymmetry on $\mathbf{D}^n$. Therefore, majorization is partial ordered, or poset on $\mathbf{D}^n$. And M(b) is called *principal majorization ideal*.

Now, we formulate our first OWA optimization problem

$$\max \sum_{j=1}^{n} w_j x_j \quad s.\ t. \quad x \in M(b). \tag{3.2}$$

Note by denition, the above problem can also be described as

$$\max \sum_{j=1}^{n} w_j x_{[j]} \quad s.\ t.\ \ x \in M(b). \tag{3.3}$$

By symmetry, the above problem is equivalent to

$$\max \sum_{j=1}^{n} w_j x_{[j]} \quad s.\ t.\ \ x \preceq b. \tag{3.4}$$

We restrict the problem on M(b) because it is convenient when consider the structure of optimal solution.

The optimization problem w is trivial if it is *Schur-convex*, i.e., $w^T x \leq w^T y$ whenever $x \preceq y$, b is an optimal solution.

Note also that optimization problem

$$\max \ \sum_{j=1}^{n} w_j x_j \quad \text{s. t.} \ \ x \preceq b \tag{3.5}$$

can be solved by greedy algorithm. It is also the result of submodular property ([4]). If $w_1 \geq w_2 \geq \cdots \geq w_n$, above problems are coincided, the solution is b_π,which is a permutation of b such that $b_1 \geq b_2 \geq \cdots \geq b_n$.

G. Dahl ([2]) investigated the extreme points structure of *M(b)*. To avoid some technicals, *b* is assumed to satisfy $b_1 > b_2 > \cdots > b_n$. Note *M(b)* is determined by two sets of inequalities, the majorization inequalities (2.1) and (2.2), ordered inequalities (2.4). The elements$(x_1, x_2, \cdots, x_n)$ of extreme point x is partitioned into some ordered blocks, blocks are separated when majorization inequalities take equalities. Within each block, ordered inequalities take equalities, i.e., elements in each block have same value.

The following notation is the weight average on a block having indexes from i to j $(1 \leq i \leq j \leq n)$ of an extreme point in M(b),

$$\hat{w}_{i,j} = (1/(j-i+1)) \sum_{k=i}^{j} w_k \sum_{r=i}^{j} b_r. \tag{3.6}$$

A dynamic programming is also given (here w are arbitrary weights, i.e., we do not need that $w_i \geq w_j$ whenever $i < j$):

Dynamic Algorithm for *M(b)*

1. Let $\mu_0 = 0$.
2. For $k = 1, 2, \cdots, n$ calculate $\mu_k = \max\{\mu_{t-1} + \hat{w}_{t,k} \ : \ t = 1, 2, \cdots, k\}$.

μ_n is the required optimization value. (The above DP is given in [2], we correct some misprints.)

Note also that if $w_1 \leq w_2 \leq \cdots \leq w_n$, i.e., Schur concave function, only one block, the optimal solution is: all components are equal to $1/n \sum_{j=1}^{n} b_j$.

4. Majorization for partially ordered sets

Recently more general partially ordered sets related to maorization has been researched ([1]). We first give the definition and review the main results.

Consider a partially set (poset) $(P, \leq_P)$ on a set with *n* elements. A ***linear extension*** of *P* is a linear order $\leq_L$ (on *P*) such that if $i \leq_P j$ then $i \leq_L j$. An ***ideal*** of poset $(P, \leq_P)$is a set $I \subseteq P$ such that $a \in I$ and $b \leq_P a$ implies $b \in I$. And a real-valued function f defined on *P* is *P-monotone* if

$$f(i) \geq f(j) \quad \text{whenever } i \leq_P j. \tag{4.1}$$

For two *P*-monotone functions f and g defined on *P*, f is *P*-majorized by g if

$$\sum_{i \in I} f(i) \leq \sum_{i \in I} g(j) \quad \text{for each ideal } I \text{ in } (P, \leq_P) \tag{4.2}$$

with equality when $I = P$. We then write $f \preceq_P g$. We should point out that $f \preceq_P g$, provided that there exists permutations π and π' of *P* such that $f_\pi \preceq_P g_{\pi'}$, here f_π is the composition $f \circ \pi$.

Theorem 4.1([1]): *Assume that* $f, g: P \rightarrow \mathbf{R}$, *satisfy* $f \preceq_P g$. *Then* $f \preceq g$ *(classical majorization) holds. However, the converse does not hold in general.*

Example 4.1: The following figure shows an examaple with poset *P*={1, 2, 3, 4}, and $1 \leq_P 2$, $1 \leq_P 3$, $2 \leq_P 4$, $3 \leq_P 4$. And the values of $f, f' (= f_\pi)$and g are defined as in Figure 4.1. It is clear that all f, f' and g are monotone. Also $f'(1)=10 \leq g(1)=10$, $f'(1)+f'(2)=10+8 \leq g(1)+g(2)=10+9$, in the same way, $10+10 \leq 10+10$, $10+8+10 \leq 10+9+10$, $10+8+10+1=10+9+10+0$. Therefore, $f' \preceq_P g$. Since f' is a permutation of f, we also have $f \preceq_P g$. By above theorem, we have $f \preceq g$ and $f' \preceq g$.

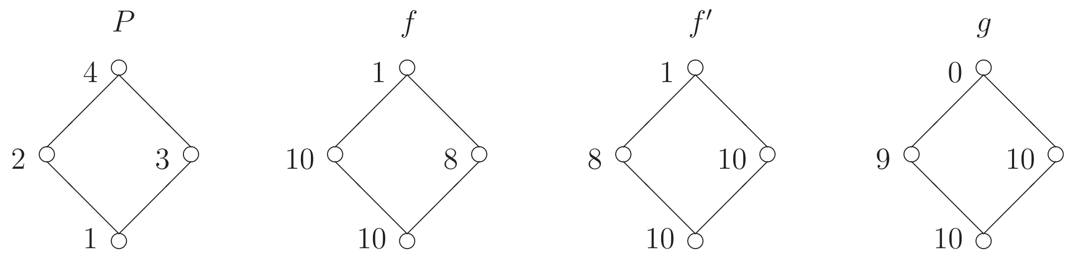

Figure 4.1. The Hasse diagram of *P*, and corresponding $f \preceq_P g$

To have a more comprehensive understanding of *P*-majorization, a simple example is given in [1]. If *P* consists two incomparable elements, and let $f = (1; 1)$; $g = (2; 0)$. Then $f \preceq g$, but $f \not\preceq_P g$. Therefore, compare classical majorization, *P*-majorization is much strong condition. If all elements are incomparable, then *P*-majorization forces all elements of both functions taking same value.

Now we show OWA optimization problem related to majorization for partially ordered sets.

Let $\mathbf{R}_P^n$ denote the set of *P*-monotone vectors in $\mathbf{R}^n$ and $b = (b_1, b_2, \cdots, b_n) \in \mathbf{R}_P^n$, the set of *P*-monotone vectors that are *P*-majorizated by b is ([1]),

$$PM(b) = \{x \in \mathbf{R}_P^n : \ x \preceq_P b\}. \tag{4.3}$$

Now, let us consider following optimization problems.

$$\max \ \sum_{j=1}^{n} w_j x_j \qquad s.\ t.\ \ x \in PM(b). \tag{4.4}$$

Although, the vertex set (4.3) has been characterized in [1], within each block, the value is not simple average, also no weighted average as in the case of principal majorization ideal. Therefore for general problem (4.4), developing an efficient algorithm seems difficulty.

As mentioned above, if all elements of *P* are incomparable, *PM*(*b*) reduced to one point. Therefore, the structure of *P* plays one key point on the complexity of (4.4).

The property of w also affects the complexity of problem. If w is Schur-convex, by Theorem 4.1, *b* is the optimal solution.

What exactly Schur-convex function means in partially ordered set, the difference between $w \in PM(b)$ and *P*-monotone w. What exactly permutation or symmetry means here, or at what condition, problem (4.4) coincides with problems (3.3) and (3.4). Of course, generally, the value of (3.4) is larger than (4.4).

5. General ordered weighted problems

As mentioned in Introduction, values in problems titled with OWA vary without symmetry. Paper [3] begins an example as follows.

Example 5.1: Consider

$$Q = \{x \in \{0,1\}^3 : \ x_1 + x_2 + x_3 = 2\}, \qquad C = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 1 & 3 \\ 5 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \omega = (1 \ \ 2 \ \ 4).$$

Let $y = Cx$, and π be permutation such that $y_{\pi\,1} \geq \cdots \geq y_{\pi\,n}$ with an appropriate index *n*. Here w is parameter of objective function for ordered values. Table 5.1 illustrates for each feasible $x \in Q$, the corresponding y, y_π and the values of OWA = ωy_π.

Table 5.1: Solutions x, values $y = Cx$, sorted values y_π and ωy_π

x	y	y_π	ωy_π=OWA$_{(C,\omega)}$
(1 1 0)	(5 2 6)	(6 5 2)	24
(1 0 1)	(2 4 7)	(7 4 2)	23
(0 1 1)	(5 4 3)	(5 4 3)	25

The OWA optimization Problem (OWAP) is

$$\text{OWAP}: \ \min_{x\in Q} \text{OWA}_{(C,\omega)}(x) = \min_{x\in Q} \ \omega(Cx)_\pi. \tag{5.1}$$

The above OWAP is an integral programing problem, and it has been shown that its complexity is *NP*-hard in general ([3]). The OWAP related to OM problem is also given in ([3]). OWAP and OM have applications in allocation and other fields ([3]). Note in Example 5.1, y is not symmetric with permutation of elements x.

A well known result in majorization is $Px \preceq x$ if P is *double stochastic matrix*, i.e., all elements are non-negative and the summation of each row and column is equal to 1. What happens if C of OWAP is double stochastic matrix will be our next works.

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