Weak Absolute Majorization Ordering and its Applications

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Abstract

Majorization is a classic topic in mathematics and applied mathematics ([4]). In this manuscript, we review some main results in majorization and their generalized format. Furthermore, we discuss some possible applications in economics.

1. Introduction

For any $p = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$, let

$$p_{[1]} \geq p_{[2]}, \cdots, \geq p_{[n]} \tag{1.1}$$

denote the components of p in decreasing order. Then, for $p, q \in \mathbf{R}^n$, if

$$\begin{cases} \sum_{i=1}^{r} p_i \leq \sum_{i=1}^{r} q_i, \ r = 1, 2, \cdots, n-1 \\ \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i \end{cases} \tag{1.2}$$

p is said to be *majorized* by q, and we write $p \prec q$.

An $n \times n$ matrix $P = p_{ij}$ is *double stochastic* if

$$p_{ij} \geq 0 \quad \text{for} \quad i, j = 1, 2, \cdots, n, \tag{1.3}$$

and

$$\sum_i p_{ij} = 1, \quad j = 1, 2, \cdots, n; \qquad \sum_j p_{ij} = 1, \quad i = 1, 2, \cdots, n; \tag{1.4}$$

Now we have two main results about double stochastic matrix and majorization.

Theorem 1.1 (Birkhoff, 1946): *The permutation matrices constitute the extreme points of the set of double stochastic matrices. Moreover, the set of double stochastic matrices is the convex hull of the permutation matrices.*

Theorem 1.2 (Hardy, Littlewood, and Pólya, 1929): *A necessary and sufficient condition that*

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$p \prec q$ is that there exist a double stochastic matrix P such that $p=qP$.

Combine the above two theorems, i.e.,

$$x = qP = q \sum_{i=1}^{n} \lambda_i Q_i = \sum_{i=1}^{n} \lambda_i q Q_i, \tag{1.5}$$

where, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda = 1$, and Q_i is a permutation matrix $(i = 1, 2, \cdots, n)$, we have

Theorem 1.3 (Rado, 1952): *The set $\{x : x \prec q\}$ is the convex hull of points obtained by permuting the components of q.*

G. Dahl and F. Margot generalized the majorization concept and show some results related to optimization problems and polytopes ([2]). Based on Dahl and Margot's result, the author introduced an absolute format ([5]). In the following, we discuss what the above main results look like in these generalized formats and also give some possible applications.

The manuscript is organized as follows. In Section 2, we give some definitions generalized concepts introduced in this section, and also discuss what the main results will look like. In Section 3, optimization problems related are presented. Several application examples are given in last section.

2. Weak Absolute Majorization Ordering

p is said to be *weakly submajorized* by q and denoted by $p \prec_w q$ if we rewrite the last equation into $\sum_{j=1}^{n} p_{[j]} \leq \sum_{j=1}^{n} q_{[j]}$ in (1.2). We say that x is *weakly k-majorized* by q and write $x \prec_k q$ if

$$\sum_{j=1}^{n} x_{[j]} \leq \sum_{j=1}^{r} q_{[j]} \quad \text{for} \quad r = 1, 2, \cdots, k. \tag{2.1}$$

Note, $p \prec_w q$ and $p \prec_n q$ are the same here.

The author extend the components of x in (2.1) to their absolute values as follows ([5]):

$$\sum_{j=1}^{r} |x_{[j]}| \leq \sum_{j=1}^{r} q_{[j]} \quad \text{for} \quad r = 1, 2, \cdots, k. \tag{2.2}$$

We say that x is *weakly absolutely k-majorized* by q and write $x_{\text{abs}} \prec_k q$. Also, we say that x is *absolutely majorized* by q and write $x_{\text{abs}} \prec q$ if $k = n-1$ in (2.2) and $\sum_{j=1}^{n} |x_{[j]}| = \sum_{j=1}^{n} q_{[j]}$.

Hereafter, we assume that *majorant* $q \in \mathbf{R}^k$ satisfies

$$q_1 \geq q_2 \geq \cdots \geq q_k \geq 0. \tag{2.3}$$

For $p \in \mathbf{R}^n$, $|p| = (\varepsilon_1 p_1, \varepsilon_2 p_2, \cdots, \varepsilon_n p_n)$ can be written as pD, where $D = \text{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ and $\varepsilon_i \in \{-1, 1\}$ $(i = 1, 2, \cdots, n)$.

A natural way to extend Theorem 1.2 is

$$pD = qPD. \tag{2.4}$$

Since $P = \sum_{i=1}^{n} \lambda_i Q_i$, where, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda = 1$, and Q_i is permutation matrix $(i = 1, 2, \cdots, n)$, we have

$$PD = \sum_{i=1}^{n} \lambda_i Q_i D. \tag{2.5}$$

For convenience, we call PD here *absolute double stochastic* matrix and $Q_iD(i = 1, 2, \cdots, n)$ *signed permutation matrices.*

When the sign of elements of diagnal matrix is not fixed, the convex combination of matrices multiplied by double stochastic matrices P_i and signed permutation matrices $D_i(i = 1, 2)$, i.e.,

$$P' = \lambda_1 P_1 D_1 + \lambda_2 P_2 D_2, \tag{2.6}$$

is not absolute double stochastic again. But, the following inequalities are satisfied,

$$\sum_i |p'_{ij}| \leq 1, \quad j = 1, 2, \cdots, n; \quad \sum_j |p'_{ij}| \leq 1, \quad i = 1, 2, \cdots, n, \tag{2.7}$$

we call it *absolute double substochastic.* For *double substochastic* matrix defined by,

$$p_{ij} \geq 0 \quad \text{for} \quad i, j = 1, 2, \cdots, n, \tag{2.8}$$

$$\sum_i p_{ij} \leq 1, \ \ j = 1, 2, \cdots, n; \quad \sum_i p_{ij} \leq 1, \ \ i = 1, 2, \cdots, n; \tag{2.9}$$

the following theorem is known:

Theorem 2.1 (Mirsky, 1959): *The set of $n \times n$ double substochastic matrices is the convex hull of the set of $n \times n$ matrices which have at most one unit in each row and each column, and all other entries are zero.*

From the above theorem, we conjecture:

The set of $n \times n$ absolute double substochastic matrices is the convex hull of the set of $n \times n$ signed permutation matrices.

And also we conjecture:

$p_{abs} \prec {}_n q$ if and only if that there exist a absolute double substochastic matrix P' such that $p = qP'$.

For absolute majorization, we are confronted with the same basic problem. In general, the set of x absolutely majorized by q, i.e., $\{x \mid x_{abs} \prec q\}$ is not convex, and no interesting results are obtainable. Therefore, *absolute* and *weak* are combined in the case of majorization.

The following result about absolute weak majorization (defined in a little different format) is known.

Proposition 2.2 (Markus, 1964): *For $x \in \mathbf{R}^n$ the set $\{x \mid x_{abs} \prec {}_n q\}$ is the convex hull of points of the form $(\varepsilon_1 q_{\pi_1}, \varepsilon_2 q_{\pi_2}, \cdots, \varepsilon_n q_{\pi_n})$, where π is a permutation and each ε_i is -1 or 1.*

The above proposition can also be obtained form the result of [5] and extreme point theorem about the system of bisubmodular function. For details about bisubmodular function, see [3].

A real-valued function f on $\mathbf{R}^n$ is *Schur convex* if $f(q) \geq f(p)$ whenever q majorizes p. Since convex property are required to obtain the main results related to Schur convex, these can not be applied to absolute case.

3. Optimization Problems

Let $c \in \mathbf{R}^n$ and consider the following optimization problem,

$$\max\{c^T x \mid x_{abs} \prec_k q\}. \tag{3.1}$$

Let

$$Ps(q;k) := \{x \in \mathbf{R}^n \mid x_{abs} \prec_k q\}. \tag{3.2}$$

Then we have ([5])

$$Ps(q;k) = \{x \in \mathbf{R}^n \mid x(X) - x(Y) \leq q(N_r) \text{ for all } (X, Y) \in 3^{\mathbf{N}_n} \text{ with } r = |X \cup Y| \leq k\}. \tag{3.3}$$

Hence $Ps(q;k)$ is also a polyhedron, or more precisely, a polytope. We know from (3.3) that the above problem is equivalent to

$$\max\{c^T x \mid x \in Ps(q;k)\}, \tag{3.4}$$

a linear programming (LP). And it is clear that when $k = n$, i.e., in the bisubmodular case, the above LP can be solved by greedy algorithm [3].

For $k < n$ the author extended a result of [2] and give a theorem about solution for (3.4). Before introducing the theorem, we define some notations.

First, for $g \in \mathbf{R}^k$, we define *tail average* of g by $\bar{g}_{s:k} := 1/(k-s+1)\sum_{i=s}^{k} |g_i|$.

Suppose that $c \in \mathbf{R}^k$ satisfies $|c_1| \geq |c_2| \geq \cdots \geq |c_{k-1}|$ (note that c_k may be arbitrary). Then there is an $m \in \{1, 2, \cdots, k\}$ such that [1]

$$\bar{c}_{1:k} \geq \bar{c}_{m:k} \leq \bar{c}_{m+1:k} \leq \cdots \leq \bar{c}_{k:k} = |c_k|. \tag{3.5}$$

For $0 \leq s \leq k-1$ and $c \in \mathbf{R}^n$ with $|c_1| \geq |c_2| \geq \cdots \geq |c_n|$, we define *signed sth q-average* $w^s \in \mathbf{R}^n$ by

$$w^s := ((-1)^{i_1} q_1, \cdots, (-1)^{i_s} q_s, (-1)^{i_{s+1}} \bar{q}_{s+1:k,} \cdots, (-1)^{i_n} \bar{q}_{s+1,k}), \tag{3.6}$$

where $i_l = 0$ or 1 for $l = 1, 2, \cdots, n$, and define $w_c^s \in \mathbf{R}^n$, *signed sth q-average related to c* by

$$w_c^s := (sign(c_1) q_1, \cdots, sign(c_s) q_s, sign(c_{s+1}) \bar{q}_{s+1:k}, \cdots, sign(c_n) \bar{q}_{s+1,k}). \tag{3.7}$$

Theorem 3.1: *The optimal solution of Problem* (3.4) *can be obtained as a permutation of* w_c^s, *where* $s = m-1$.

The concept of majorization and weakly absolute k-majorization can be applied to finance, e.g. to control the risk of investment trusts or funds. As many other linear programming, the problem of (3.4) can also be applied to high dividend investment trusts. Recently, because historically low interest of bank deposit and Internet service, finance investment is very popular. In next section, we give some details of them.

4. Application Examples

Most (stock) investment trusts do not open their portfolios, to author's knowledge, Sawakami investment trust opens the details of its portfolio every month. So, our examples are based on the portfolios of Sawakami investment trust.

First, Table 1 gives the recent portfolio of top thirty of Sawakami investment trust (we omit here the name of corporations). And Figure 1 is the accumulation curve based on the data of the table.

Table 1 Portfolio of Sawakami investment trust (2006/10/31)

No.	Portfolio	cumulation	No.	Portfolio	cumulation	No.	Portfolio	cumulation
1	1.26	1.26	11	0.74	9.72	21	0.64	16.46
2	1.16	2.42	12	0.73	10.45	22	0.64	17.10
3	0.96	3.38	13	0.72	11.17	23	0.63	17.73
4	0.88	4.26	14	0.70	11.87	24	0.62	18.35
5	0.86	5.12	15	0.68	12.55	25	0.60	18.96
6	0.82	5.94	16	0.66	13.21	26	0.60	19.55
7	0.80	6.74	17	0.66	13.87	27	0.60	20.15
8	0.76	7.50	18	0.65	14.52	28	0.59	20.74
9	0.74	8.24	19	0.65	15.17	29	0.59	21.33
10	0.74	8.96	20	0.65	15.82	30	0.59	21.92

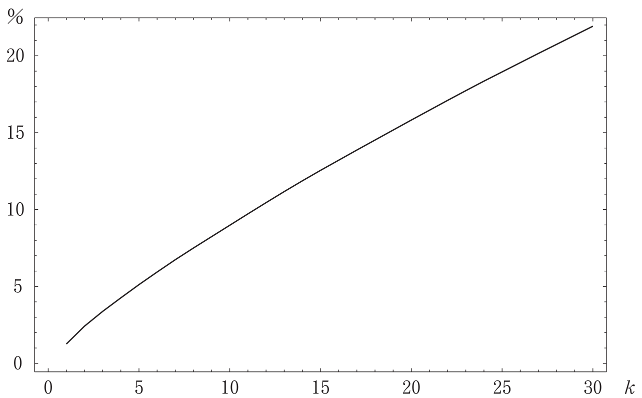

Figure 1 portfolio of top thirty

If a curve lies below the curve in Figure 1, we can say that the risk is more evenly distributed. Such discussion is something like the Lorenz curve ([4]). Note portfolio is just one way of the argument. We may also consider the fluctuation of stock prices in a period. When high dividend is considered, as recent boom, we can apply the contents in Section 3. In the following we give some examples further.

Figure 2 gives a curve calculated in following step:

(1) multiply the fluctuation of stock prices in six months by their portfolios,

(2) rearrange these values in deceasing order,

(3) accumulate them.

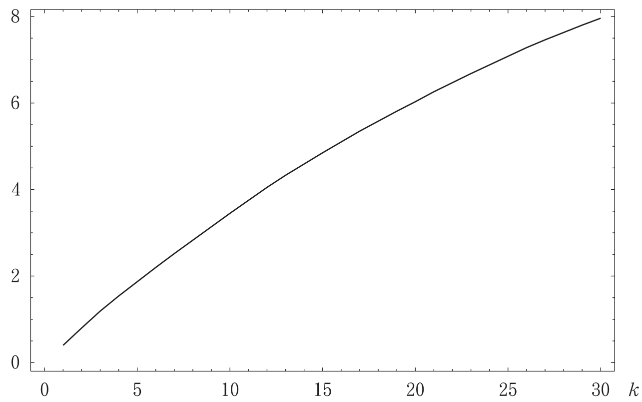

Figure 2 fluctuation of top thirty

The curve given in Figure 3 is calculated in the same way as the curve in Figure 2, except for the step (1), where the fluctuation versus current stock price is considered.

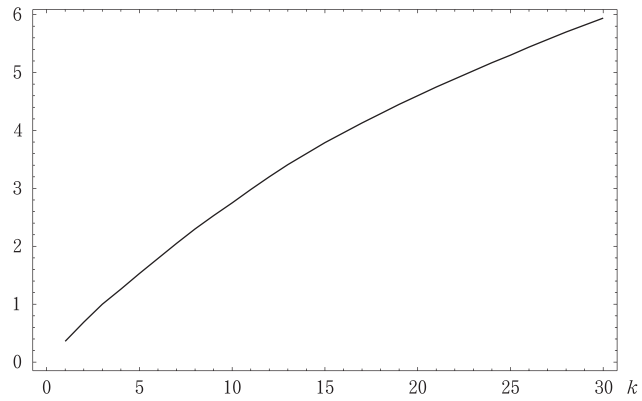

Figure 3 fluctuation versus current prices of top thirty

For high dividend investment trusts, we increase the portfolios of high dividend stocks, while keep the curve under the curves given in above figures. If one of the curve is considered as restriction condition, optimal solution (portfolio) can be obtained according to results in Section 3.

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